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A Gibbs Phenomenon for Spline Functions

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DEDICATED TO THE MEMORY OF I. J. SCHOENBERG

An analogue of the Gibbs phenomenon is shown to hold for approximation by periodic spline functions on uniform subdivisions.

1. INTRODUCTION

In the nineteenth century it was observed that if a function f has a jump discontinuity, the partial sums of its Fourier series are not sufficiently close to f near the jump; they tend to overshoot f by a factor proportional to the jump. In 1899, J. Willard Gibbs [1] gave a mathematical description of the phenomenon that now bears his name.

Although Gibbs looked at a sawtooth function, his result may be illustrated with a square wave. Let

$$F(x)=\begin{cases} -1, & -1\leqslant x<0 \\ 1, & 0\leqslant x\leqslant 1 \end{cases} \tag{1.1}$$

and suppose $\tau_n=\tau_n(F,x)$ is the trigonometric polynomial of the form $a_0/2+\sum_{r=1}^{n}(a_r\cos \pi rx+b_r\sin \pi rx)$ that best approximates F in the norm of $L^2[-1,1]$. (See Fig. 1.) Then

$$\lim_{n\to\infty}\tau_n\left(\frac{x}{n}\right)=\tau(x)=\frac{2}{\pi}\int_0^{\pi x}\left(\frac{\sin t}{t}\right)dt \qquad \text{locally uniformly in } x. \tag{1.2}$$

Setting $x=1$ yields the overshoot characteristic of the Gibbs phenomenon:

$$\lim_{n\to\infty}\tau_n\left(\frac{1}{n}\right)=\tau(1)=1.17898\cdots>1=F(0^+). \tag{1.3}$$

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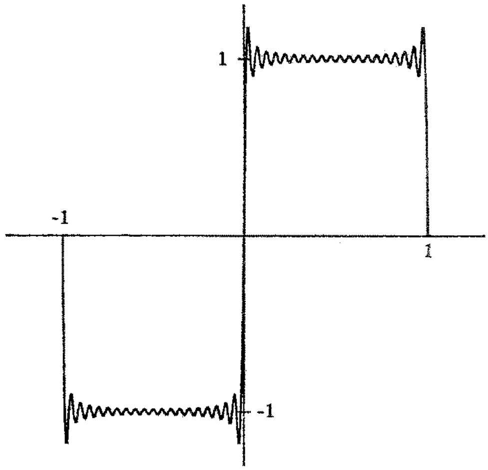

FIG. 1. $\tau_n(F, x)$.

Some years ago, I. J. Schoenberg conjectured to the author that an exact analogue of (1.2) should hold for spline functions. The present paper shows this to be the case for periodic splines with equal knot spacing.

Let $T_n^{[k]}(x)$ be the periodic spline of degree $k-1$ having knots

$$\left\{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, ..., 1-\frac{1}{n}, 1\right\} \tag{1.4}$$

that best approximates F in the norm of $L^2[-1, 1]$. It is shown in Section 3 that there exists a spline $S^{[k]}(x)$ of degree $k-1$ with knots at the integers such that $\lim_{n\to\infty} T_n^{[k]}(x/n) = S^{[k]}(x)$ locally uniformly in x.

The functions $S^{[k]}(x)$, $k \leq 8$, are numerically computed in Section 4. If $F(0^-) \neq F(0^+)$, it is shown that they behave very much like the Gibbs limiting function $\tau(x)$; in particular, an overshoot is always observed.

The results of Section 3 are used in Section 5, where the behavior of $S^{[k]}(x)$ as $k \to \infty$ is investigated. In the light of previous results of Schoenberg [6] it seems natural to expect the classical trigonometric case to be approached uniformly; i.e., the Gibbs phenomenon for splines of high fixed degree should look very much like the classical Gibbs phenomenon. This suggests the following

Conjecture. Let $F(x)$ be the square wave (1.1) and $\tau(x)$ be defined as in (1.2). Then

$$\lim_{k\to\infty} S^{[k]}(x) = \tau(x) \qquad \text{uniformly in } x. \tag{1.5}$$

Strong evidence for the validity of (1.5) is contained in Section 5, where it is shown that for splines of odd degree $k-1$

$$\lim_{k \to \infty} S^{[k]}(x) = \tau(x), \qquad x = 0, \pm 1, \pm 2, \ldots. \tag{1.6}$$

The only previous result I am aware of that has the flavor of (1.2) is found in Golomb [2]. Assuming $U_n(x)$ is the periodic spline of degree $2k-1$ which interpolates to a sufficiently smooth periodic function G at the knots (1.4), he shows there exists a constant θ_k such that the $2k$th derivative of G may be calculated as the limit

$$(D^{2k}G)(0) = \theta_k \lim_{n \to \infty} n^{2k} \left[G\left(\frac{1}{2n}\right) - U_n\left(\frac{1}{2n}\right) \right].$$

Note that if $D^k G = F$, then $D^k U_n = T_n^{[k]}$.

2. Preliminary Results

We first must discuss a few elementary facts, all of which may be found in [5] or [7].

Let $\{x_i\}$ be a finite or infinite set of increasing real numbers where $a = \inf(x_i)$ and $b = \sup(x_i)$. If k is an integer $\geqslant 2$, $S(x)$ is a spline function of order k or degree $k-1$ with knots $\{x_i\}$ if

(i) The restriction of $S(x)$ to $[x_{i-1}, x_i]$ is a polynomial of degree at most $k-1$ and

(ii) $S \in C^{k-2}(a, b)$.

If in addition the number of knots is finite and

(iii) $S^{(\nu)}(a) = S^{(\nu)}(b)$, $\nu = 0, 1, \ldots, k-2$,

$S(x)$ is called a periodic spline. A spline having knots only at the integers is a cardinal spline.

Let

$$B_1(x) = \begin{cases} 1, & -\frac{1}{2} \leqslant x \leqslant \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

and define the k-fold convolution

$$B_k(x) = B_1 * B_1 * \cdots * B_1(x), \qquad k \geqslant 2 \tag{2.1}$$

and associated cardinal spline

$$\tilde{B}_k(x) = \begin{cases} B_k(x), & k \text{ even} \\ B_k(x+\frac{1}{2}), & k \text{ odd}. \end{cases} \tag{2.2}$$

These functions, called cardinal B-splines, possess the following properties:

(i) The functions $\tilde{B}_k(x-\nu)$, $\nu = 0, \pm 1, \pm 2, \ldots$, form a basis for the space of cardinal splines of order k; (2.3)

(ii) $B_k(x) \geqslant 0$ is an even function and has support on $[-k/2, k/2]$; (2.4)

(iii) $\int_{-\infty}^{\infty} \tilde{B}_k(x)\, dx = 1$; (2.5)

(iv) $\sum_{\nu=-\infty}^{\infty} \tilde{B}_k(x-\nu) = 1$, $-\infty < x < \infty$; (2.6)

(v) $B_k(x)$ has Fourier transform $\psi_k(t) = ((2 \sin(t/2))/t)^k$; (2.7)

(vi) $\int_{-\infty}^{\infty} \tilde{B}_k(x-\nu)\, \tilde{B}_k(x-\mu)\, dx = B_{2k}(\nu-\mu)$. (2.8)

For $k = 2r$ an even positive integer, we consider the rational function

$$\Phi_k(z) = \sum_{\nu=-\infty}^{\infty} B_k(\nu)\, z^\nu \tag{2.9}$$

having only simple zeros $\lambda_i = \lambda_i(k)$ satisfying

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{r-1} > -1 > \lambda_r > \cdots > \lambda_{k-2} \tag{2.10}$$

and

$$\lambda_1 \lambda_{k-2} = \lambda_2 \lambda_{k-3} = \cdots = \lambda_{r-1} \lambda_r = 1. \tag{2.11}$$

Thus $1/\Phi_k(z)$ is holomorphic in the annulus $|\lambda_{r-1}| < |z| < |\lambda_r|$ and has corrresponding Laurent expansion

$$\frac{1}{\Phi_k(z)} = \sum_{\nu=-\infty}^{\infty} \omega_\nu^{(k)} z^\nu. \tag{2.12}$$

Note that

$$\sum_{\nu=-\infty}^{\infty} |\omega_\nu^{(k)}| < \infty \tag{2.13}$$

and

$$\omega_\nu^{(k)} = \omega_{-\nu}^{(k)}, \qquad \nu = 1, 2, 3, \ldots. \tag{2.14}$$

The Wiener-Levy theorem implies the sequence convolution transformation $B_k\colon l^\infty \to l^\infty$ defined by

$$\sum_{\mu=-\infty}^{\infty} B_k(\nu-\mu)\, c_\mu = d_\nu, \qquad \nu = 0, \pm 1, \pm 2, \ldots$$

is inverted by

$$\sum_{\mu=-\infty}^{\infty} \omega_{\nu-\mu}^{(k)}\, d_\mu = c_\nu, \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{2.15}$$

We also note that

$$\phi_k(t) = \Phi_k(e^{it}) = \sum_{\nu=-\infty}^{\infty} \psi_k(t + 2\pi\nu), \qquad t \text{ real}. \tag{2.16}$$

It will also be convenient to discuss the "fundamental" spline $L_k(x)$ defined as the unique bounded cardinal spline of even order k satisfying the interpolatory conditions

$$L_k(\nu) = \begin{cases} 1, & \nu = 0 \\ 0, & \nu = \pm 1, \pm 2, \ldots. \end{cases}$$

Then for any bounded cardinal spline $S(x)$ of even order k we may write $S(x) = \sum_{\nu=-\infty}^{\infty} S(\nu)\, L_k(x-\nu)$. The fundamental spline has Fourier transform representation

$$L_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_k(t)}{\phi_k(t)}\, e^{itx}\, dt \tag{2.17}$$

and decays for fixed k

$$|L_k(x)| \to 0 \qquad \text{exponentially as } |x| \to \infty. \tag{2.18}$$

3. Existence of the Gibbs Spline

The main result of this section is the following

THEOREM 1. *Suppose $F \in L^\infty[-1, 1]$ and both $F(0^+)$ and $F(0^-)$ exist. Let $T_n = T_n^{[k]}(F, x)$ be the periodic spline function of order k having knots $\{\nu/n\}_{\nu=-n}^{n}$ that best approximates F in the norm of $L^2[-1, 1]$. Then there exists a cardinal spline $S(x) = S^{[k]}(x)$ of order k satisfying*

$$\lim_{n\to\infty} T_n\left(\frac{x}{n}\right) = S(x) \qquad \textit{locally uniformly in } x. \tag{3.1}$$

We shall refer to $S(x)$ as the Gibbs spline of order k.

For purposes of the proof, it is convenient to rescale F and T_n to $[-n, n]$, i.e., define

$$S_n(x) = T_n\left(\frac{x}{n}\right) \quad \text{and} \quad f_n(x) = F\left(\frac{x}{n}\right), \qquad -n \leqslant x \leqslant n. \tag{3.2}$$

Note that both S_n and f_n may be extended by periodicity to $(-\infty, \infty)$. Thus S_n can be viewed as a cardinal spline with period $2n$ and by (2.3) has a unique representation of the form

$$S_n(x) = \sum_{\nu=-\infty}^{\infty} c_\nu^n \tilde{B}_k(x-\nu). \tag{3.3}$$

LEMMA 1. *The sequence $\{c_\nu^n\}_{\nu=-\infty}^{\infty}$ defined by* (3.3) *satisfies the doubly infinite system of normal equations*

$$\sum_{\mu=-\infty}^{\infty} B_{2k}(\nu-\mu)\, c_\mu^n = f_\nu^n = \int_{-\infty}^{\infty} f_n(x)\, \tilde{B}_k(x-\nu)\, dx, \qquad \nu = 0, \pm 1, \pm 2, ... \tag{3.4}$$

Proof. Let $\bar{B}_k(x) = \sum_{\mu=-\infty}^{\infty} \tilde{B}_k(x - 2\mu n)$ be the central B-spline of period $2n$. Since $c_\nu^n = c_{\nu+2\mu n}^n$, $\mu = 0, \pm 1, \pm 2, ...$, by the unicity of periodic spline approximation, (3.3) implies $S_n(x) = \sum_{\nu=-n}^{n-1} c_\nu^n \bar{B}_k(x-\nu)$. But as S_n minimizes the quantity

$$\int_{-n}^{n} [S_n(x) - f_n(x)]^2\, dx = \int_{-n}^{n} \left[\sum_{\mu=-n}^{n-1} c_\mu^n\, \bar{B}_k(x-\mu) - f_n(x)\right]^2 dx$$

setting $\partial/\partial c_\nu^n = 0$ gives

$$\sum_{\mu=-n}^{n-1} \left[\int_{-n}^{n} \bar{B}_k(x-\mu)\, \bar{B}_k(x-\nu)\, dx\right] c_\mu^n$$
$$= \int_{-n}^{n} f_n(x)\, \bar{B}_k(x-\nu)\, dx, \qquad \nu = -n, -n+1, ..., n-1,$$

which after some manipulations becomes

$$\sum_{\mu=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{B}_k(x-\mu)\, \tilde{B}_k(x-\nu)\, dx\right] c_\mu^n$$
$$= \int_{-\infty}^{\infty} f_n(x)\, \tilde{B}_k(x-\nu)\, dx, \qquad \nu = 0, \pm 1, \pm 2, ...$$

Property (2.8) now yields (3.4). ∎

Inverting (3.4) by means of (2.15) implies

LEMMA 2.

$$c_\nu^n = \sum_{\mu=-\infty}^{\infty} \omega_{\nu-\mu}^{(2k)} f_\mu^n, \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{3.5}$$

Now define the function f^* and sequences $\{f_\nu\}$ and $\{c_\nu\}$ in the following manner:

$$f^*(x) = \begin{cases} F(0^+), & x \geqslant 0 \\ F(0^-), & x < 0 \end{cases} \tag{3.6}$$

$$f_\nu = \int_{-\infty}^{\infty} f^*(x)\, \tilde{B}_k(x-\nu)\, dx, \qquad \nu = 0, \pm 1, \pm 2, \ldots \tag{3.7}$$

and

$$c_\nu = \sum_{\mu=-\infty}^{\infty} \omega_{\nu-\mu}^{(2k)} f_\mu, \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{3.8}$$

It will shortly be shown that

$$S(x) = \sum_{\nu=-\infty}^{\infty} c_\nu \tilde{B}_k(x-\nu) \tag{3.9}$$

is the Gibbs spline of Theorem 1.

LEMMA 3.

$$\lim_{n\to\infty} c_\nu^n = c_\nu, \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{3.10}$$

Proof. Subtracting (3.8) from (3.5) and using (2.13) gives

$$c_\nu^n - c_\nu = \sum_{\mu=-\infty}^{\infty} \omega_{\nu-\mu}^{(2k)} [f_\mu^n - f_\mu] = \sum_{|\mu-\nu| \leqslant R} + \sum_{|\mu-\nu| > R}, \tag{3.11}$$

where R is chosen so that $\sum_{|\mu|>R} |\omega_\mu^{(2k)}| < \varepsilon$. Since $|f_\mu^n - f_\mu| \leqslant 2\, \|f\|_\infty$ by (2.5), (3.4), and (3.7), the second sum in (3.11) satisfies

$$\sum_{|\mu-\nu|>R} |\omega_{\nu-\mu}^{(2k)}|\, |f_\mu^n - f_\mu| < 2\varepsilon\, \|f\|_\infty.$$

To handle the first sum, observe that for all x such that $\nu - R - k/2 - \frac{1}{2} < x < \nu + R + k/2$ and n chosen sufficiently large $|f_n(x) - f^*(x)| = |F(x/n) - F(0^{\pm})| < \varepsilon$. Thus by properties (2.4) and (2.5)

$$|f_\mu^n - f_\mu| \leqslant \int_{-\infty}^{\infty} |f_n(x) - f^*(x)| \, \tilde{B}_k(x-\mu) \, dx \leqslant \int_{\mu - k/2 - 1/2}^{\mu + k/2}$$

$$\leqslant \int_{\nu - R - k/2 - 1/2}^{\nu + R + k/2} < \varepsilon, \qquad \mu = \nu - R, \nu - R + 1, ..., \nu + R.$$

Hence

$$\sum_{\mu = \nu - R}^{\nu + R} |\omega_{\nu-\mu}^{(2k)}| \, |f_\mu^n - f_\mu| < \varepsilon \sum_{\mu = -\infty}^{\infty} |\omega_\mu^{(2k)}|.$$

This establishes (3.10). ∎

Theorem 1 follows from Lemma 3, (2.6), (3.3), and (3.9) by noting if $|x| \leqslant$ constant, then $S(x) - S_n(x) = \sum_{\nu=-\infty}^{\infty} (c_\nu - c_\nu^n) \, \tilde{B}_k(x - \nu)$ is really a finite sum.

Before ending this section, a result of independent interest should be noted; at a jump discontinuity the best least squares approximants converge to the midpoint of the jump.

THEOREM 2. *Let F and T_n be as in Theorem . Then*

$$\lim_{n \to \infty} T_n(0) = \frac{F(0^-) + F(0\ \ )}{2}.$$

Proof. With no loss of generality assume $F(0^-) = -F(0^+)$. By Theorem 1 it is sufficient to show $S(0) = 0$. But this follows immediately by applying the symmetry conditions (2.4), (2.14), and (3.6) in (3.7), (3.8), and (3.9). ∎

4. CONSTRUCTION OF THE GIBBS SPLINE AND EXAMPLES

As yet we have not shown that a Gibbs phenomenon exists, the hallmark of which is the overshoot given in (1.3). This section investigates the behavior of the Gibbs spline $S^{[k]}(x)$ for orders $k \leqslant 8$. For each of these cases an overshoot

$$\max_{0 < x < \infty} S^{[k]}(x) > F(0^+) \tag{4.1}$$

is observed, i.e., a Gibbs phenomenon exists.

Before proceeding, we need the following

LEMMA 4. *Let Φ_{2r}, $\lambda_i = \lambda_i(2r)$, and $\omega_\nu^{(2r)}$ be defined by* (2.9), (2.10), *and* (2.12), *respectively. Then for $r \geqslant 2$*

$$\omega_\nu^{(2r)} = \sum_{i=1}^{r-1} A_i \lambda_i^{|\nu|}, \qquad \nu = 0, \pm 1, \pm 2, ..., \tag{4.2}$$

where

$$A_i = \frac{1}{\lambda_i \Phi'_{2r}(\lambda_i)}. \tag{4.3}$$

Proof. From (2.12) it follows that

$$\omega_\nu^{(2r)} = \frac{1}{2\pi i} \oint_c \frac{z^{-\nu-1}}{\Phi_{2r}(z)} dz, \qquad \nu = 0, \pm 1, \pm 2, ..., \tag{4.4}$$

where the contour of integration is the unit circle $|z| = 1$. Since $\omega_\nu = \omega_{-\nu}$, we may assume that $\nu \leqslant 0$. Then the integrand of (4.4) has poles inside $|z| = 1$ only at points λ_i with corresponding residues $\lambda_i^{|\nu|}/\lambda_i \Phi'_{2r}(\lambda_i)$, $i = 1, 2, ..., k-1$. Evaluating (4.4) by means of residues establishes the lemma. ▮

To simplify computations, the following examples are worked for the function (1.1); in particular $F(0^-) = -1$ and $F(0^+) = 1$. Note that as in the proof of Theorem 2, $S^{[k]}(x)$ is now an odd function.

EXAMPLE 1. THE PIECEWISE LINEAR CASE ($k = 2$). Since $\tilde{B}_2(x-\nu) = B_2(x-\nu)$ has positive or negative support if ν is respectively strictly positive or strictly negative and $B_2(x)$ is an even function, we have

$$f_\nu = \int_{-\infty}^{\infty} f^*(x)\, B_2(x-\nu)\, dx = \begin{cases} 1, & \nu = 1, 2, ... \\ 0, & \nu = 0 \\ -1, & \nu = -1, -2, ... \end{cases}$$

The rational function

$$\Phi_4(z) = \frac{1 + 4z + z^2}{6z}$$

has roots $\lambda = -2 + \sqrt{3}$ and λ^{-1}. Hence Lemma 4 gives

$$\omega_\nu^{(4)} = \sqrt{3}\, \lambda^{|\nu|}, \qquad \nu = 0, \pm 1, \pm 2, ...$$

Noting that $S^{[2]}(\nu)=S(\nu)=c_\nu$ for piecewise linear cardinal splines and applying (3.8) yields

$$S(\nu)=1-\lambda^\nu, \qquad \nu=0, 1, 2, \dots.$$

This completely determines $S^{[2]}(x)$; in particular

$$\max S(x)=S(1)=3-\sqrt{3}=1.2679>1=F(0^+).$$

EXAMPLE 2. THE QUADRATIC CASE ($k=3$). For the even degree case we have

$$f_\nu=\int_{-\infty}^{\infty} f^*(x)\,\tilde{B}_3(x-\nu)\,dx=\int_{-\infty}^{\infty} f^*(x)\,B_3\left(x-\nu+\frac{1}{2}\right)dx.$$

Thus

$$f_1=-f_0=\int_{-1/2}^{1/2} B_3(x)\,dx=\frac{2}{3}$$

and

$$f_\nu=\begin{cases} 1, & \nu=2, 3, \dots \\ -1, & \nu=-1, 2, \dots. \end{cases}$$

Since

$$\Phi_6(z)=\frac{1+26z+66z^2+26z^3+z^4}{120z^2}$$

has roots $\lambda_1=-0.043096$, $\lambda_2=-0.430575$, λ_1^{-1}, and λ_2^{-1}, Lemma 4 implies

$$\omega_\nu^{(6)}=A_1\lambda_1^{|\nu|}+A_2\lambda_2^{|\nu|}, \qquad \nu=0, \pm 1, \pm 2, \dots,$$

where $A_1=-0.252815$ and $A_2=3.094986$.

Equations (3.8) are used to solve for c_ν, $\nu=0, \pm 1, \pm 2, \dots$, which are then plugged into (3.9):

$$S(x)=\sum_{\nu=-\infty}^{\infty} c_\nu B_3\left(x-\nu+\frac{1}{2}\right).$$

TABLE I

k	α_1	$S^{[k]}(\alpha_1)$	β_1	$S^{[k]}(\beta_1)$	α_2	$S^{[k]}(\alpha_2)$	β_2	$S^{[k]}(\beta_2)$
2	1.0000	1.2680	2.0000	0.9282	3.0000	1.0192	4.0000	0.9948
3	0.8234	1.1974	1.7209	0.9174	2.7014	1.0358	3.6993	0.9845
4	1.1502	1.2042	2.1707	0.8935	3.1739	1.0568	4.1746	0.9696
5	0.8690	1.1861	1.7802	0.8994	2.7327	1.0609	3.7139	0.9628
6	1.1256	1.1923	2.1785	0.8904	3.1964	1.0707	4.2028	0.9535
7	0.9019	1.1834	1.8244	0.8977	2.7735	1.0683	3.7443	0.9524
8	1.0952	1.1899	2.1569	0.8905	3.1881	1.0753	4.2033	0.9458
τ	1.0000	1.1790	2.0000	0.9028	3.0000	1.0662	4.0000	0.9499

Letting $\alpha_i, \beta_i, i=1, 2, ...,$ be respectively the positive local maxima and minima of $S(x)$ in increasing order, we numerically calculate

$$\alpha_1 = 0.8234, \qquad S(\alpha_1) = 1.1974$$

$$\beta_1 = 1.7209, \qquad S(\beta_1) = 0.9174$$

$$\alpha_2 = 2.7014, \qquad S(\alpha_2) = 1.0358$$

$$\beta_2 = 3.6993, \qquad S(\beta_2) = 0.9845.$$

Corresponding values of $S^{[k]}(x), 2 \leqslant k \leqslant 8$, are presented in Table I, together with the Gibbs function $\tau(x)$. All of these functions (except $S^{[2]}(x)$) have graphs as in Fig. 2, which is actually the graph of $S^{[6]}(x)$. In each case the value of the overshoot corresponding to (1.3) is

$$S^{[k]}(\alpha_1) = \max_{0 < x < +\infty} S^{[k]}(x) > 1. \tag{4.5}$$

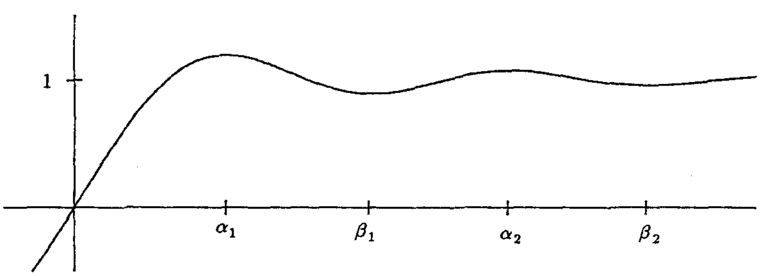

FIG. 2. $S^{[k]}(x)$.

5. The Gibbs Spline as the Degree Approaches Infinity

The remainder of the paper is devoted to examining the behavior of the Gibbs spline $S^{[k]}(x)$ as $k \to \infty$. The central result is

THEOREM 3. *Let k be an even integer and $F(x)$ the square wave defined by* (1.1). *Then*

$$\lim_{k \to \infty} S^{[k]}(x) = \frac{2}{\pi} \int_0^{\pi x} \left(\frac{\sin t}{t} \right) dt, \qquad x = 0, \pm 1, \pm 2, \ldots. \tag{5.1}$$

The following result of Schoenberg [8] is crucial:

LEMMA 5. *Let $L_k(x)$ be the fundamental cardinal spline of even order k. Then*

$$\lim_{k \to \infty} L_k(x) = \frac{\sin \pi x}{\pi x} \qquad \textit{uniformly in } x. \tag{5.2}$$

This suggests using the basis

$$S^{[k]}(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_k(x-\nu), \qquad y_\nu = y_\nu^{[k]} = S^{[k]}(\nu). \tag{5.3}$$

Hence (5.1) can be obtained by observing the behavior of $y_\nu^{[k]}$ as $k \to \infty$. Since $S^{[k]}(x)$ is now an odd function, it is only necessary to prove (5.1) for x a positive integer.

LEMMA 6. *The sequence $(y_\nu) \in l^\infty$ defined by* (5.3) *satisfies*

$$\sum_{\mu=-\infty}^{\infty} \Lambda(\nu-\mu)\, y_\mu = f_\nu, \qquad \nu = 0, \pm 1, \pm 2, \ldots, \tag{5.4}$$

where

$$f_\nu = \int_{-\infty}^{\infty} f^*(x)\, L_k(x-\nu)\, dx \tag{5.5}$$

$$f^*(x) = \begin{cases} 1, & x \geqslant 0 \\ -1, & x < 0 \end{cases} \tag{5.6}$$

and

$$\Lambda(\nu-\mu) = \int_{-\infty}^{\infty} L_k(x-\nu)\, L_k(x-\mu)\, dx. \tag{5.7}$$

Proof. Recall from Lemma 1 of Section 3 the fact that $S_n(x)$ is a best approximation to $f_n(x)$; it follows that

$$\int_{-\infty}^{\infty} S_n(x)\, \tilde{B}_k(x-v)\, dx = \int_{-\infty}^{\infty} f_n(x)\, \tilde{B}_k(x-v)\, dx,$$

which if we let $n \to \infty$ and use (2.4), (3.1), and (3.2) becomes

$$\int_{-\infty}^{\infty} S^{[k]}(x)\, \tilde{B}_k(x-v)\, dx = \int_{-\infty}^{\infty} f^*(x)\, \tilde{B}_k(x-v)\, dx.$$

Since by (2.18), $L_k(x) = \sum_{\mu=-\infty}^{\infty} c_\mu \tilde{B}_k(x-\mu)$ with exponentially decaying coefficients (c_μ) we obtain

$$\int_{-\infty}^{\infty} S^{[k]}(x)\, L_k(x-v)\, dx = \int_{-\infty}^{\infty} f^*(x)\, L_k(x-v)\, dx,$$

which with (5.3) yields (5.4). ▮

The representation (2.17) allows us to write

$$\Lambda(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\psi_k(t)}{\phi_k(t)} \right)^2 e^{ivt}\, dt, \qquad v = 0, \pm 1, \pm 2, \ldots,$$

which by virtue of (2.7) and (2.16) becomes

$$\Lambda(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\phi_{2k}(t)}{\phi_k^2(t)}\, e^{ivt}\, dt, \qquad v = 0, \pm 1, \pm 2, \ldots. \tag{5.8}$$

Using (2.10) we see that $\phi_{2k}(t)/\phi_k^2(t) \geqslant c > 0$ for all real t. Hence by applying the Wiener–Levy theorem as in Section 2 the sequence convolution transformation (5.4) is inverted by

$$\sum_{\mu=-\infty}^{\infty} \Omega(v-\mu)\, f_\mu = y_v, \qquad v = 0, \pm 1, \pm 2, \ldots, \tag{5.9}$$

where the sequence $\Omega = \Omega^{[k]}(v)$ satisfies

$$\Omega(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\phi_k^2(t)}{\phi_{2k}(t)}\, e^{ivt}\, dt, \qquad v = 0, \pm 1, \pm 2, \ldots \tag{5.10}$$

$$\sum_{v=-\infty}^{\infty} |\Omega(v)| < \infty \tag{5.11}$$

and

$$\Omega(v) = \Omega(-v), \qquad v = 1, 2, 3, \ldots. \tag{5.12}$$

It is now necessary to discuss some asymptotic properties of $\Omega^{[k]}(\nu)$ as $|\nu| \to \infty$. The following estimates are quite similar to those obtained by Schoenberg [7] for the approximation properties of exponential Euler splines.

LEMMA 7. *If $k \geqslant 2$ and $-\pi \leqslant t \leqslant \pi$, then*

$$\left| \frac{\phi_k^2(t)}{\phi_{2k}(t)} - 1 \right| \leqslant \frac{\pi^2}{2} \left(\frac{|t|}{\pi} \right)^k. \tag{5.13}$$

Proof.

$$\begin{aligned} \frac{\phi_k^2(t)}{\phi_{2k}(t)} - 1 &= \frac{\phi_k^2(t) - \phi_{2k}(t)}{\phi_{2k}(t)} \\ &= \frac{1}{\phi_{2k}(t)} \sum_{\substack{\mu, \nu = -\infty \\ \mu \neq \nu}}^{\infty} \psi_k(t + 2\pi\mu)\, \psi_k(t + 2\pi\nu) \end{aligned}$$

and since $\phi_{2k}(t) = \sum_{\nu=-\infty}^{\infty} \psi_{2k}(t + 2\pi\nu)$ is a sum of non-negative terms, we have

$$\left| \frac{\phi_k^2(t)}{\phi_{2k}(t)} - 1 \right| \leqslant \sum_{\substack{\mu, \nu = -\infty \\ \mu \neq \nu}}^{\infty} \left| \frac{\psi_k(t + 2\pi\mu)\, \psi_k(t + 2\pi\nu)}{\psi_{2k}(t)} \right|. \tag{5.14}$$

Let $[\mu, \nu]$ denote the (μ, ν)th term of the above series. If both $\mu, \nu \neq 0$ and $t \in [-\pi, \pi]$, then $|t + 2\pi\nu| = \pi\, |t/\pi + 2\nu| \geqslant \pi(2\,|\nu| - 1)$ and

$$[\mu, \nu] = \left| \frac{t^{2k}}{(t + 2\pi\mu)^k\, (t + 2\pi\nu)^k} \right| \leqslant \left(\frac{|t|}{\pi} \right)^{2k} \frac{1}{(2\,|\mu| - 1)^k\, (2\,|\nu| - 1)^k}. \tag{5.15}$$

If, say, $\mu = 0$ then $\nu \neq 0$ and for $t \in [-\pi, \pi]$

$$[\mu, \nu] = \left| \frac{t^k}{(t + 2\pi\nu)^k} \right| \leqslant \left(\frac{|t|}{\pi} \right)^k \frac{1}{(2\,|\nu| - 1)^k}. \tag{5.16}$$

Combining (5.15) and (5.16) and using $k \geqslant 2$ gives

$$[\mu, \nu] \leqslant \left(\frac{|t|}{\pi} \right)^k \frac{1}{(2\,|\mu| - 1)^2\, (2\,|\nu| - 1)^2},$$

which when used in (5.14) implies (5.13). ∎

Since by (5.10)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\phi_k^2(t)}{\phi_{2k}(t)} - 1 \right) e^{i\nu t}\, dt = \begin{cases} \Omega(\nu), & \nu = \pm 1, \pm 2, \ldots \\ \Omega(0) - 1, & \nu = 0 \end{cases} \tag{5.17}$$

the symmetry of the above integrand, Parseval's identity, and the previous lemma yield

$$(\Omega(0)-1)^2+\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}(\Omega(\nu))^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\phi_k^2(t)}{\phi_{2k}(t)}-1\right]^2 dt$$

$$\leqslant\frac{1}{2\pi}\left(\frac{\pi^2}{2}\right)^2\int_{-\pi}^{\pi}\left(\frac{t}{\pi}\right)^{2k}dt\leqslant\frac{\pi^4}{4(2k+1)}.$$

This establishes the following

LEMMA 8. *Let* $\Omega=\Omega^{[k]}(\nu)$ *be as defined in* (5.9). *Then*

$$\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}(\Omega(\nu))^2\leqslant\frac{\pi^4}{4(2k+1)} \tag{5.18}$$

and

$$|\Omega(0)-1|\leqslant\frac{\pi^2}{2\sqrt{2k+1}}. \tag{5.19}$$

It should be noted that the better estimate

$$|\Omega(0)-1|\leqslant\frac{\pi^2}{2(k+1)}$$

may be obtained by setting $\nu=0$ in (5.17).

Returning to (5.9), we see by (5.11) and (5.12) that

$$S^{[k]}(\nu)=y_\nu=\Omega(0)f_\nu+\sum_{\mu=1}^{\infty}\Omega(\mu)(f_{\nu-\mu}+f_{\nu+\mu}),\qquad \nu=0,\pm 1,\pm 2,\dots \tag{5.20}$$

This representation provides the motivation for the following lemmas.

LEMMA 9. *The sequence* (f_ν) *defined by* (1.1), (5.4), *and* (5.5) *is given by*

$$f_\nu=2\operatorname{sgn}(\nu)\int_0^{|\nu|}L_k(x)\,dx,\qquad \nu=0,\pm 1,\pm 2,\dots \tag{5.21}$$

The proof is an easy consequence of (5.5), (5.6), and the symmetry of $L_k(x)$.

In view of the sum in (5.20) and the symmetry of $S^{[k]}(x)$ it is only necessary to bound $|f_{\nu-\mu}+f_{\nu+\mu}|$ for $\mu,\nu\geqslant 1$.

LEMMA 10.

$$\sum_{\mu=1}^{\infty} (f_{\nu-\mu}+f_{\nu+\mu})^2 \leqslant 24\nu^2, \qquad \nu=1, 2, 3, \ldots \tag{5.22}$$

Proof. First suppose $\mu \leqslant \nu$. Then $\nu-\mu \geqslant 0$ and by (5.21) and the Cauchy–Schwarz inequality

$$\begin{aligned} |f_{\nu-\mu}+f_{\nu+\mu}| &\leqslant 2\left[\int_0^{\nu-\mu} |L_k(x)|\, dx + \int_0^{\nu+\mu} |L_k(x)|\, dx\right] \\ &\leqslant 4\int_0^{2\nu} |L_k(x)|\, dx \\ &\leqslant 4(2\nu)^{1/2}\left[\int_0^{2\nu} (L_k(x))^2\, dx\right]^{1/2}, \qquad \mu=1, 2, \ldots, \nu. \end{aligned}$$

Similarly if $\mu > \nu$

$$\begin{aligned} |f_{\nu-\mu}+f_{\nu+\mu}| &= \left|-2\int_0^{\mu-\nu} L_k(x)\, dx + 2\int_0^{\mu+\nu} L_k(x)\, dx\right| \\ &\leqslant 2\int_{\mu-\nu}^{\mu+\nu} |L_k(x)|\, dx \\ &\leqslant 2(2\nu)^{1/2}\left[\int_{\mu-\nu}^{\mu+\nu} (L_k(x))^2\, dx\right]^{1/2}, \qquad \mu=\nu+1, \nu+2, \ldots. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{\mu=1}^{\infty} [f_{\nu-\mu}+f_{\nu+\mu}]^2 &\leqslant \nu\cdot 16\cdot 2\nu\int_0^{\infty} (L_k(x))^2\, dx + 4\cdot 2\nu\cdot 2\nu\int_0^{\infty} (L_k(x))^2\, dx \\ &\leqslant 48\nu^2\int_0^{\infty} (L_k(x))^2\, dx, \qquad \nu=1, 2, 3, \ldots. \end{aligned} \tag{5.23}$$

In [3], it is shown that

LEMMA 11.

$$\int_{-\infty}^{\infty} (L_k(x))^2\, dx < 1, \qquad k=2r \geqslant 2. \tag{5.24}$$

Applying (5.24) to (5.23) yields (5.22). ∎

We are finally in a position to prove Theorem 3. It is enough to show

$$\lim_{k \to \infty} S^{[k]}(\nu) = 2 \int_0^{\nu} \frac{\sin(\pi x)}{\pi x}\, dx, \qquad \nu = 1, 2, 3, ... \tag{5.25}$$

Using (5.20) we have

$$S^{[k]}(\nu) - 2 \int_0^{\nu} \frac{\sin(\pi x)}{\pi x}\, dx = (\Omega(0) - 1) f_\nu + \left(f_\nu - 2 \int_0^{\nu} \frac{\sin(\pi x)}{\pi x}\, dx \right) + \sum_{\mu=1}^{\infty} \Omega(\mu)(f_{\nu-\mu} + f_{\nu+\mu}). \tag{5.26}$$

Applying the four previous lemmas to the first and third terms on the right gives

$$|(\Omega(0) - 1) f_\nu| \leqslant \frac{\pi^2}{2\sqrt{2k+1}} \cdot 2 \int_0^{\nu} |L_k(x)|\, dx \leqslant \frac{\pi^2 \sqrt{\nu}}{\sqrt{2k+1}} \left(\int_0^{\infty} (L_k(x))^2\, dx \right)^{1/2} < \frac{\pi^2 \sqrt{\nu}}{\sqrt{2(2k+1)}}$$

and

$$\left| \sum_{\mu=1}^{\infty} \Omega(\mu)(f_{\nu-\mu} + f_{\nu+\mu}) \right| \leqslant \left(\sum_{\mu=1}^{\infty} (\Omega(\mu))^2 \right)^{1/2} \left(\sum_{\mu=1}^{\infty} (f_{\nu-\mu} + f_{\nu+\mu})^2 \right)^{1/2} \leqslant \left(\frac{\pi^4}{8(2k+1)} \cdot 24\nu^2 \right)^{1/2}$$

Letting $k \to \infty$ in (5.26) and using (5.2) and (5.21) on the middle term yields (5.25). ∎

In closing, it should be mentioned that representations other than (5.3) are possible and may prove useful in investigating the conjecture (1.5). One such is the following "wavelet" decomposition of $S^{[k]}(x)$.

Define the exponentially decaying sequence

$$\alpha_K^{[k]}(-\nu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\nu t}}{\sqrt{\phi_{2k}(t)}}\, dt, \qquad \nu = 0, \pm 1, \pm 2, ...$$

and corresponding cardinal spline

$$K_k(x) = \sum_{\nu=-\infty}^{\infty} \alpha_K^{[k]}(\nu)\, \tilde{B}_k(x - \nu).$$

Then the sequence $\{K_k(x-v)\}_{v=-\infty}^{\infty}$ forms an orthonormal basis in L_2 for the space of cardinal splines of order k (see [4]). Thus due to the exponential decay of $K_k(x)$, every function f of at most power growth has a K_k-cardinal series representation $\sum_{v=-\infty}^{\infty} a_v K_k(x-v)$, where $a_v = \int_{-\infty}^{\infty} f(t)\, K_k(t-v)\, dt$.

From the application of this to f^* and use of the argument of Lemma 6 it follows that the Gibbs spline has K_k-cardinal series expansion

$$S^{[k]}(x) = \sum_{v=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f^*(t)\, K_k(t-v)\, dt \right) K_k(x-v).$$

REFERENCES

1. J. W. GIBBS, Letter to the Editor, *Nature* **59** (April 27, 1899), 606.
2. M. GOLOMB, Approximation by periodic spline interpolants on uniform meshes, *J. Approx. Theory* **1** (1968), 26–65.
3. F. B. RICHARDS, Uniform spline interpolation operators in L_2, *Illinois J. Math.* **18** (1974), 516–521.
4. S. D. RIEMENSCHNEIDER AND Z. SHEN, Box splines, cardinal series, and wavelets, *in* "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), Academic Press, Boston, 1990.
5. I. J. SCHOENBERG, Cardinal interpolation and spline functions, *J. Approx. Theory* **2** (1969), 167–206.
6. I. J. SCHOENBERG, Notes on spline functions. I. The limits of the interpolating periodic spline functions as their degree tends to infinity, *Indag. Math.* **34** (1972), 412–422.
7. I. J. SCHOENBERG, "Cardinal Spline Interpolation," Regional Conference Monograph No. 12, SIAM, Philadelphia, 1973.
8. I. J. SCHOENBERG, Notes on spline functions. III. On the convergence of the interpolating cardinal splines as their degree tends to infinity, *Israel J. Math.* **16** (1973), 87–83.